TWISTOR DIAGRAMS for Yang-Mills scattering amplitudes

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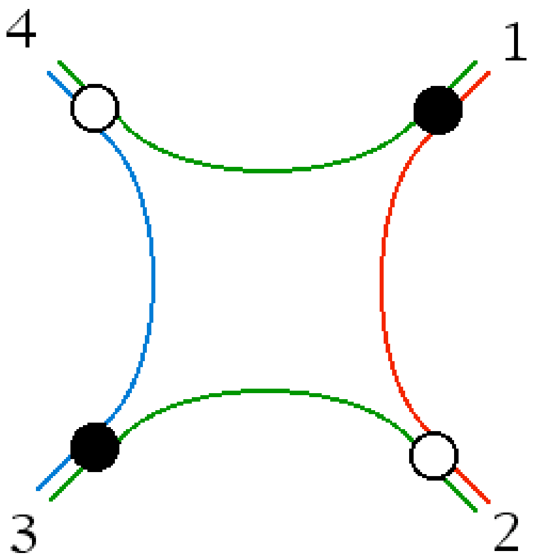


The tree-level scattering amplitude for four gluons, expressed in twistors. It looks like a string...

- Roger Penrose's *twistor programme:* twistor space is primary.
- Develop space-time-inspired methods.
- Conformal symmetry(-breaking) is fundamental, left-right symmetry is not.
- Free massless fields from holomorphic homogeneous cohomology elements in one twistor variable.
- **Off-shell fields from two twistor variables.**
- **Twistor-geometric regularization.**

Contour integration in several dimensions

$$\frac{1}{2\pi i}\oint f(z)\frac{dz}{z-q} = f(q)$$

Residue theorem...

$$\frac{1}{2\pi i}\oint \frac{f_{-1}(\pi^{A'})\, D\pi}{\pi^{A'}\alpha_{A'}} = f_{-1}(\alpha^{A'})$$

CP^1 version

$$\frac{1}{2\pi i}\oint \frac{D\pi \wedge D\sigma}{(\pi^{A'}\sigma_{A'})^2} = 1$$

NEW type of contour:
S^2 in $CP^1 \times CP^1 - CP^1$

$$\frac{1}{(2\pi i)^3}\oint \frac{D^2\pi \wedge D^2\sigma}{(\pi^{A'}\sigma_{A'})^2} = 1$$

Non-projective version:
$S^1 \times S^3$

$$\frac{1}{(2\pi i)^3}\oint \frac{D^2\pi \wedge D^2\sigma}{(\pi^{A'}\sigma_{A'} - k)^2} = 1$$

A generalisation:
result is independent of k

Develop two-dimensional contours by deformation...

$$\frac{1}{(2\pi i)^2}\oint_{\pi.\sigma=0} \frac{1}{(\pi^{A'}\alpha_{A'})^2}\frac{1}{(\sigma^{B'}\beta_{B'})^2}\, D\pi \wedge D\sigma = \frac{1}{(\alpha_{A'}\beta^{A'})^2}$$

Contour with topology of disc,
circular *boundary.*

From spinor to twistor integrals

Generalise all this to CP^n. The *boundary* contours are fundamental. No analogue in one-dimensional contour integration. They have the effect of taking *anti-derivatives.*

Bracket factors $[\theta]_n$ defined so that $d/d\theta\, [\theta]_n = [\theta]_{n+1}$

$[\theta]_{-1}$ means a boundary on $\theta = 0$.

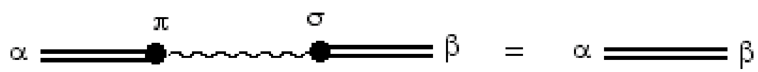
$[\theta]_0$ means $1/\theta$, with an S^1 contour round $\theta = 0$

$[\theta]_n$ means $(-1)^n\, n!/\theta^{n+1}$, with S^1 contour round $\theta = 0$

and $[\theta]_{-n-1}$ means $\theta^n/n!$ with boundary on $\theta = 0$.

Graphical elements: single line for simple pole, double line for double pole and so on... and wavy line for boundaries.

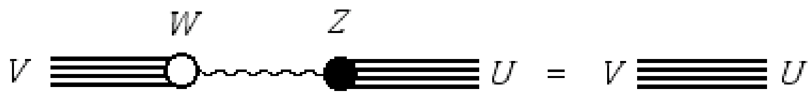
The spinor integral with boundary is:



Analogous twistor integral is:

$$\oint_{W_\alpha Z^\alpha = 0} \frac{\Gamma(4)}{(V^\alpha W_\alpha)^4} \frac{\Gamma(4)}{(Z^\alpha U_\alpha)^4} DW \wedge DZ = \frac{\Gamma(4)}{(V^\alpha U_\alpha)^4}$$

Use black vertex for twistor variable, white vertex for dual twistor variable.



These give the elements of *twistor diagrams*: the simplest possible elements in algebraic geometry.

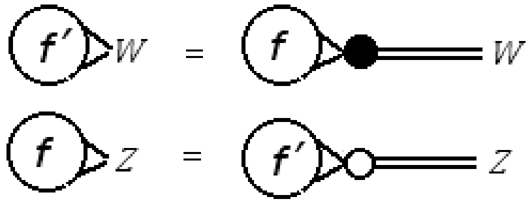
Free-field twistor diagrams

A free massless field of helicity n/2 can be represented by 1-functions

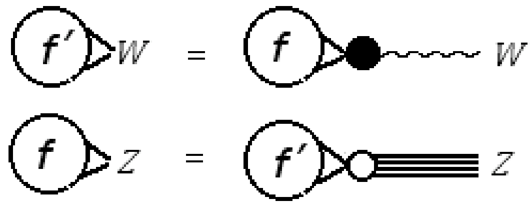
- $f_{-n-2}(Z^{\alpha})$ of homogeneity degree $(-n-2)$
- $f'_{n-2}(W_{\alpha})$ of homogeneity degree $(n-2)$.

These are connected by *twistor transforms*, defined by these twistor diagram elements.

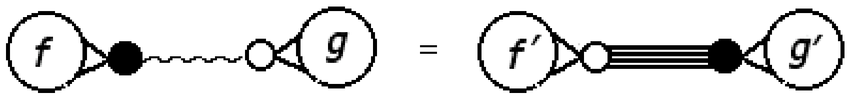
For n=0, scalar fields, the 1-functions of degree (− 2) are related by:



For n=1, gauge fields, the 1-functions of degree 0 and (− 4) are related by:



The inner product between two gauge fields f and g, giving the zeroth order scattering amplitude (for no interaction):

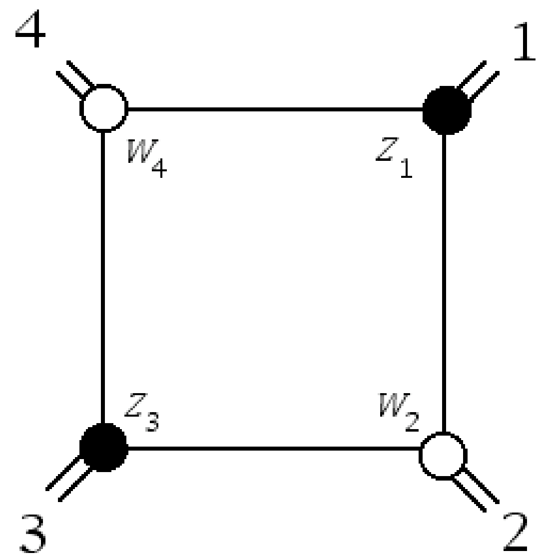


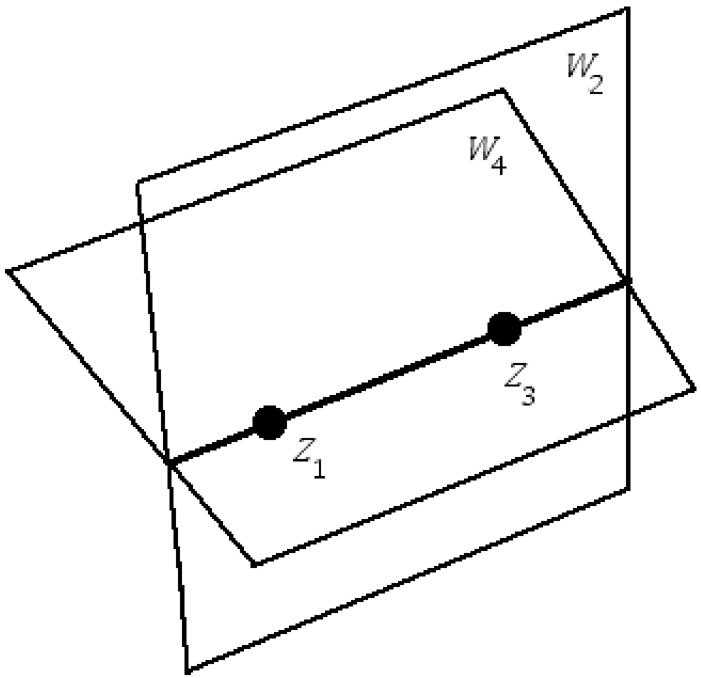
- simple
- finite
- conformally invariant
- model for scattering amplitudes.

Four scalar fields

Massless **scalar** fields are represented by functions of degree (−2). The product of four such fields, integrated over space-time, gives the starting-point for all scattering amplitudes.

Guiding idea: The four simple poles are incidence relations which constrain Z_1, W_2, Z_3, W_4 to lie on a common line, which is a point in complexified Minkowski space.





Describe contour by:

$$Z_1^\alpha = (ix^{AA'}\pi_{A'}, \pi_{A'}),$$

$$Z_3^\alpha = (ix^{AA'}\sigma_{A'}, \sigma_{A'})$$

Integrate over W_2, W_4, π and σ. The result is:

$$\phi_1(x)\,\phi_2(x)\,\phi_3(x)\,\phi_4(x)$$

This leaves 4 more dimensions in the integration, to produce:

$$\int \phi_1(x)\,\phi_2(x)\,\phi_3(x)\,\phi_4(x)\;d^4x$$

where the contour runs over a copy of Minkowski space.

Many scalar fields

The 'box' twistor diagram represents

$$1\,.\,\delta(\sum_{i=1}^{4} p_i)$$

(Momentum states not needed.)

Twistor representations of

$$1\,.\,\delta(\sum_{i=1}^{n} p_i)$$

are found by
an extension of this argument.

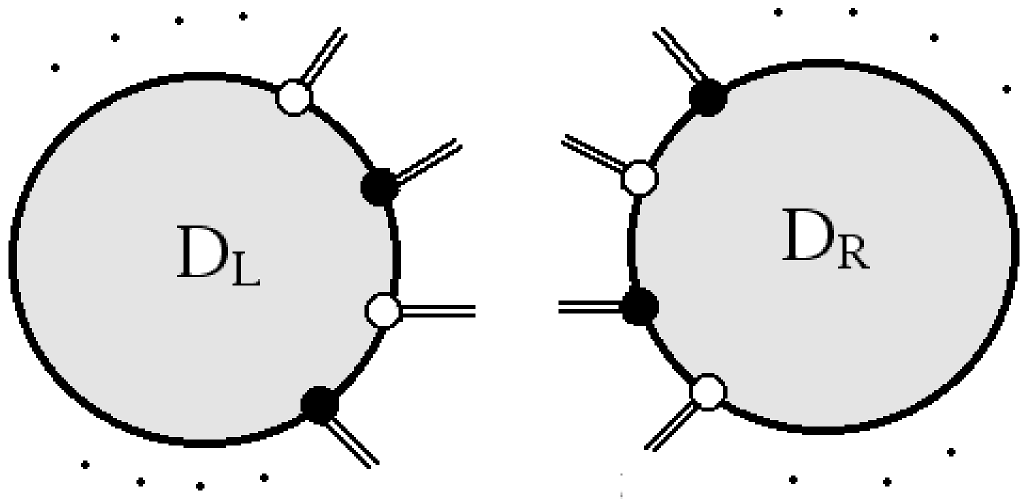
Conformal-symmetry-breaking numerator factors are needed.

$$I_{\alpha\beta}X^{\alpha}Z^{\beta},\;\; I^{\alpha\beta}W_{\alpha}Y_{\beta}$$

are represented by dashed lines

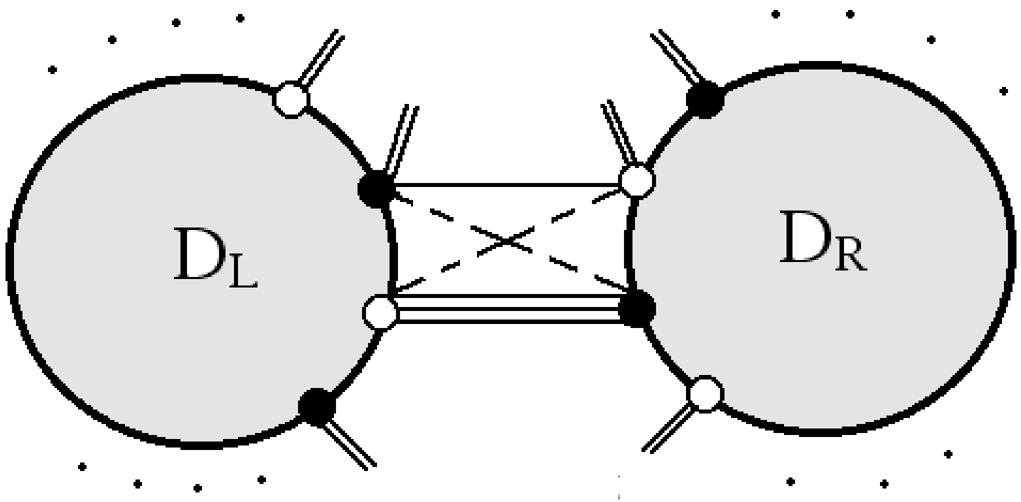
- - - - - - - - - -

Suppose



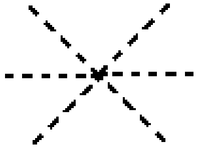
are twistor diagrams for m and n scalar fields.

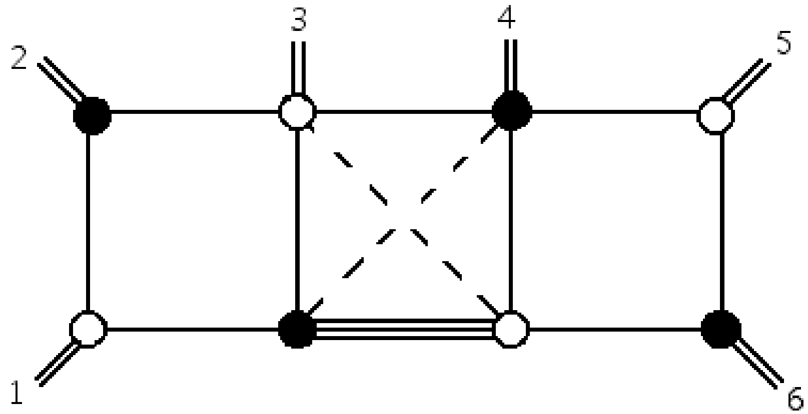
Then

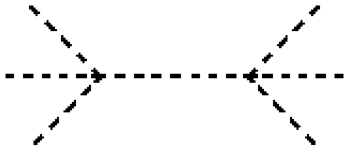


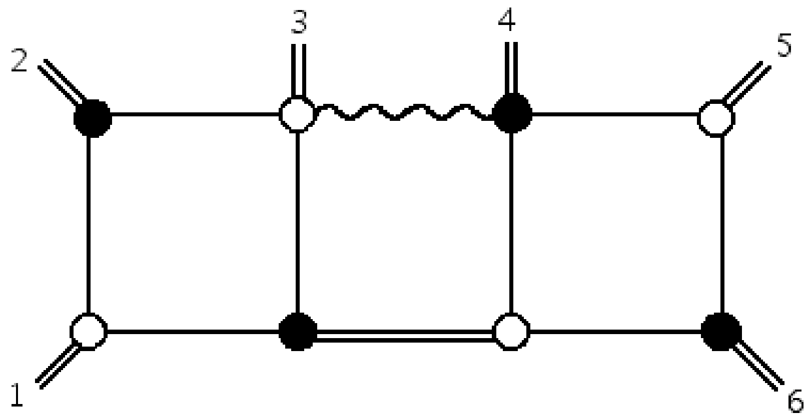
is a twistor diagram for (m+n − 2) scalar fields.

Application to φ⁴ theory









$$\frac{1}{S_{123}}\delta\Big(\sum_{i=1}^{6} p_i\Big), \qquad S_{123} = (p_1 + p_2 + p_3)^2$$

The integrand is conformally invariant. But contours will break the conformal invariance, by having boundaries at the infinity of Minkowski space.

In fact, there are *no* contours in projective twistor space, because of infra-red divergence.

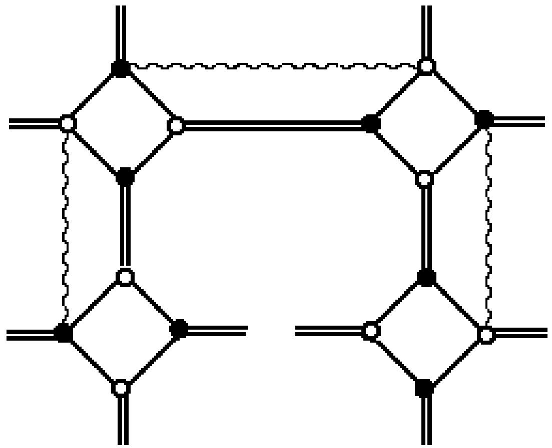
Twistor regularisation: exploit the extra dimension of twistor scale in non-projective twistor space. Change:

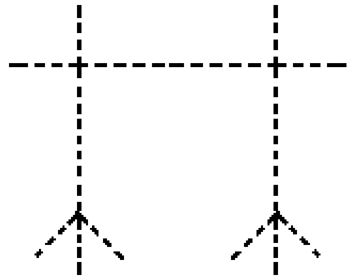
$$W_\alpha Z^\alpha \longrightarrow W_\alpha Z^\alpha - k$$

$$I_{\alpha\beta} X^\alpha Z^\beta \longrightarrow I_{\alpha\beta} X^\alpha Z^\beta - m, \qquad I^{\alpha\beta} W_\alpha Y_\beta \longrightarrow I^{\alpha\beta} W_\alpha Y_\beta - m$$

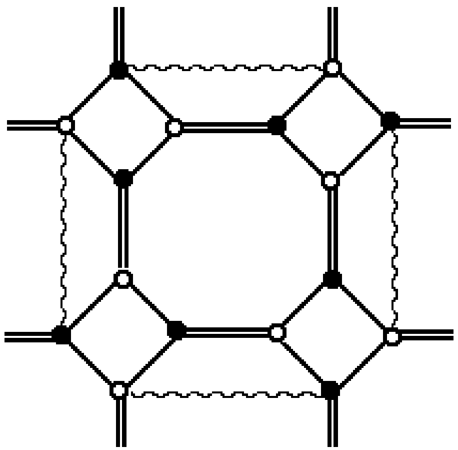
Scalar 1-loop integral

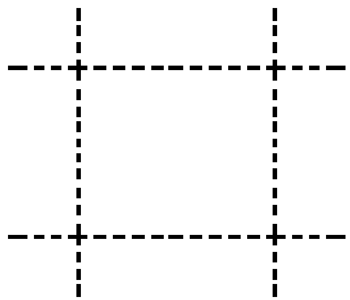
Similarly:





and by an extension of the argument,

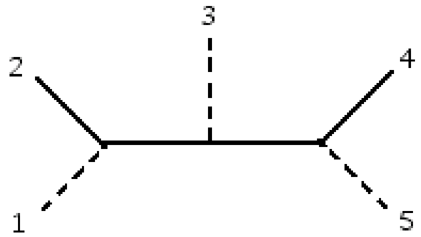




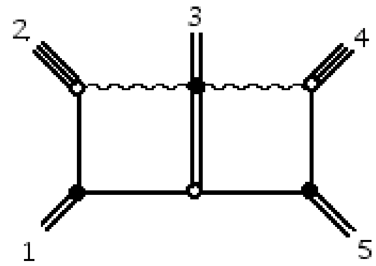
This gives the full 1-loop 'box function' in '4-mass' case. For null momenta at the vertices, change from 2-twistor functions to 1-twistor functions. This is natural via Yukawa interaction diagrams.

Massless Yukawa interaction theory

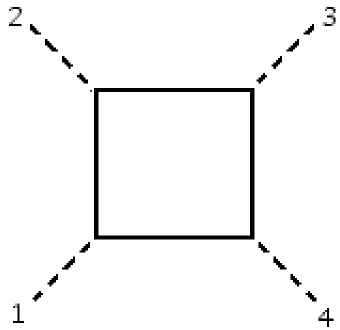
ALL vertices and edges in Yukawa Feynman diagrams can be translated into twistor diagrams.



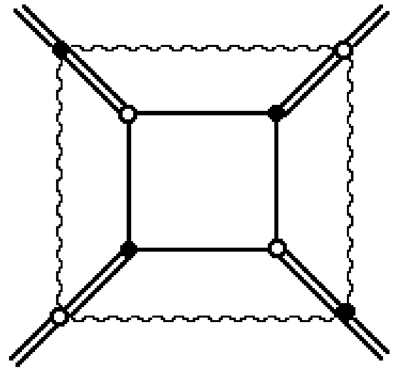
$$\frac{\langle 15 \rangle}{\langle 12 \rangle \langle 45 \rangle} \delta(\sum_{i=1}^{5} p_i)$$



By a similar argument, the '0-mass' case arises as:



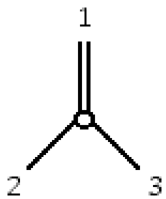
$$\frac{1}{2} \ln^2(S_{12}/S_{23})\, \delta(\sum_{i=1}^{4} p_i)$$



To be studied/checked.

The intermediate cases (1-mass, 2-mass, 3-mass) can be treated similarly.

Moreover formal 3-amplitude is simple:



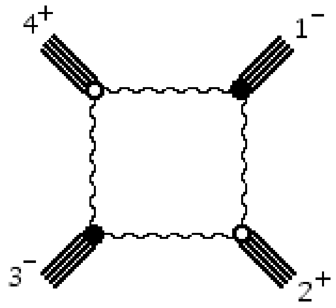
This is the simplest theory in which to understand twistor diagram structure.

But gauge theory offers the chance for twistor diagrams to do better than Feynman diagrams.

Pure gauge-field scattering

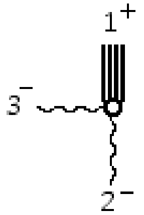
1970: Roger Penrose noted that for Compton scattering, both Feynman diagrams are absorbed into one gauge-invariant twistor diagram.

1990: AH noted string-geometric look of pure-gauge diagram



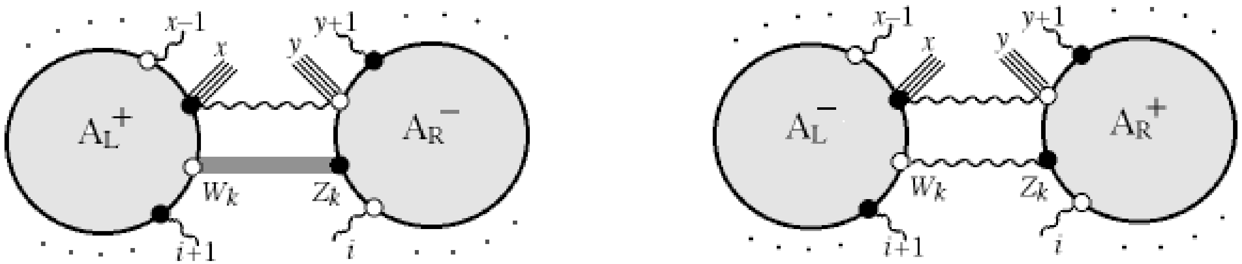
$$\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,\delta\Big(\sum_{i=1}^{4} p_i\Big)$$

2005: Formal 3-amplitude is:



$$\frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle}\,\delta\Big(\sum_{i=1}^{3} p_i\Big)$$

2005: **arxiv.org/hep-th/0503060.** Britto-Cachazo-Feng-Witten recursion identified as a rule for joining twistor diagrams for the sub-amplitudes: a summation over terms of form



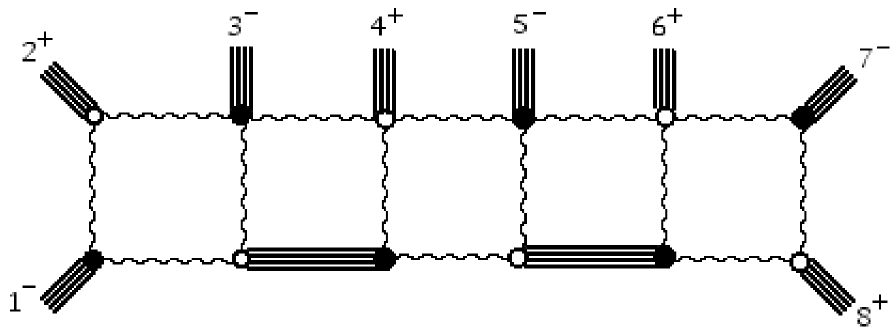
'Off-shell' momenta appear as *two-twistor* functions.

Practical application

All tree amplitudes can be written as twistor diagrams involving only boundaries and quadruple poles.

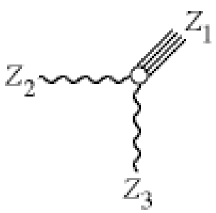
This graph-theoretic formalism is of practical value for streamlining the complicated algebra of BCF.

Example: One of the 20 terms for $A(1^-2^+3^-4^+5^-6^+7^-8^+)$, as evaluated by Britto Cachazo and Feng in 2004:

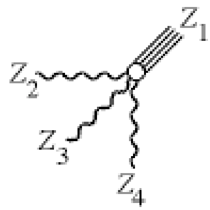


$$\frac{\langle 13\rangle^4[68]^4\langle 5|(6+7+8)|4]^4}{S_{123}S_{678}\langle 12\rangle\langle 23\rangle[67][78]\langle 1|2+3|4][8|6+7|5\rangle[6|(7+8)(1+2+3)|4]\langle 5|(6+7+8)(1+2)|3\rangle}\,\delta\Big(\sum_{i=1}^{8} p_i\Big)$$

Collinearity and coplanarity of twistors, as defined by Witten, can be read off from these diagrams since



Collinear twistors



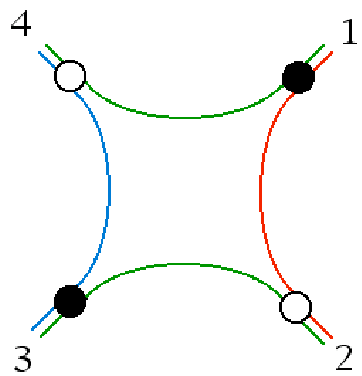
Coplanar twistors

Symmetries and singularity structures are transparent.

Super-diagrams

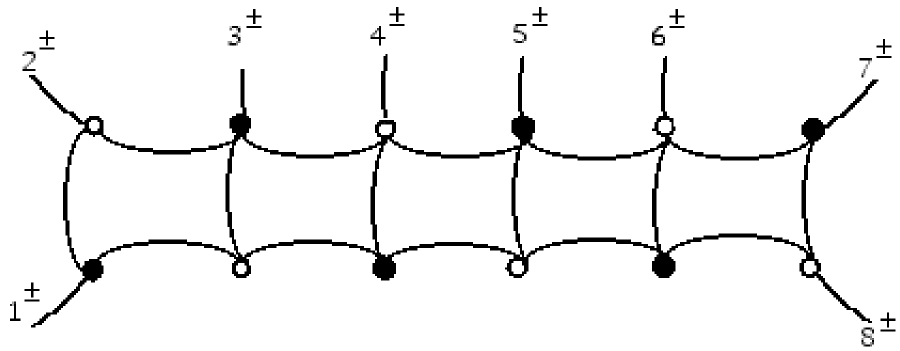
Eliminate the quadruple poles in favour of all-boundary diagrams, i.e. pure geometry?
Yes, if we define formal *super-boundaries* by changing W.Z to W.Z + φ.ψ, where φ and ψ are N=4 anticommuting variables. The BCFW rule becomes simpler and its restriction on helicities can be dropped.
arxiv.org/hep-th/0512336.

Draw the super-boundaries as *arcs*. The formal 3-amplitudes arise as three arcs meeting at a vertex.
The complete 4-field tree-level diagram for $A(1^{\pm}2^{\pm}3^{\pm}4^{\pm})$ is



There is an alternative twistor-geometric approach which doesn't use supersymmetry.

Applied practically to efficient evaluation of all 8-field tree amplitudes for the helicity-conserved (NNMHV) sector. Still 20 diagrams, but each one can be evaluated for all 256 cases at once (70 of them non-trivial). Typically:



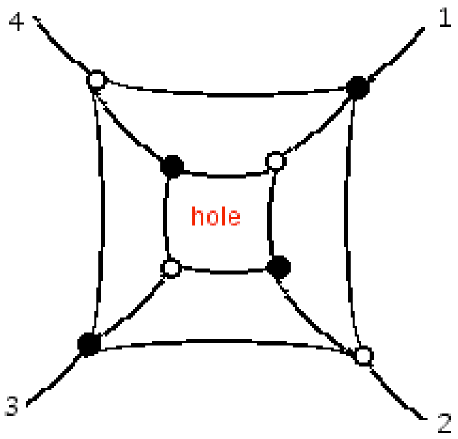
$$\frac{F(H)^4}{S_{123}S_{678}\langle 12\rangle\langle 23\rangle[67][78]\langle 1|2+3|4][8|6+7|5\rangle[6|(7+8)(1+2+3)|4]\langle 5|(6+7+8)(1+2)|3\rangle}\,\delta(\sum_{i=1}^{8} p_i)$$

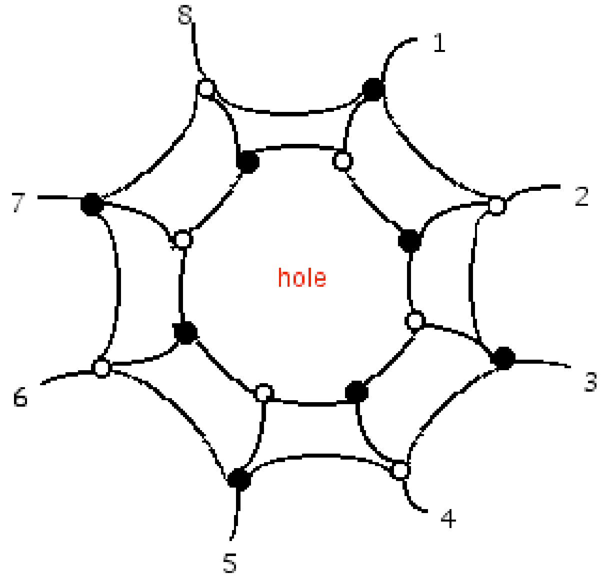
where the 70 F(H) are read off from the diagrams and tabulated in **arxiv.org/hep-th/0603101**

One-loop amplitudes

Put this tree-level structure together with the singularity structure found in scalar/Yukawa theories.

Conjecture: one-loop N=4 Super-Yang-Mills, conserved helicity sector, for four fields and eight fields:





Conjecture: that the simplicities of the one-loop N=4 Yang-Mills structure

- box functions are enough
- No UV, simple IR
- cut constructibility...

all flow from this structure.

Conjecture: there must be a string-like generating rule formulated directly in twistor space.

Conjecture: Gravity can be treated in the same way, with a simple 3-amplitude and an N=8 analogue.