String amplitudes and twistor diagrams: an analogy

A. P. Hodges*
Mathematical Institute, Oxford University
24–29 St. Giles, Oxford OX1 3LB, U.K.

A proposal has been made [1, 2, 3] for a conformal field theory in four dimensions (CFT4), in which the *twistor* representation of quantum fields plays an essential role. According to this proposal, a scattering amplitude could arise in the first instance as associated with a specific complex manifold X (bearing a specified relationship to flat twistor space); to obtain a physically meaningful amplitude a summation of these amplitudes would be performed over (a class of) such manifolds. It was noted that physically meaningful amplitudes (for theories of interacting massless fields in Minkowski space-time) have already appeared in twistor theory. They have arisen in the formalism of *twistor diagrams* [4, 5, 6] — compact contour integrals in products of twistor spaces, with a rough analogy to Feynman diagrams, but with certain features similar to the dual diagrams of bosonic string theory. It was therefore suggested that there might exist a direct connection between the CFT4 picture and the twistor diagram formalism. This speculation is here strengthened by noting an analogy between standard two-dimensional string theory and a specific twistor-diagrammatic calculation, suggesting that the integration of twistor diagrams could be interpreted as a summation over complex manifolds.

We consider the derivation of the Veneziano amplitude for four open strings of spin 0. According to the standard theory [7, page 49], the amplitude associated with one string is determined by mapping that string conformally onto a disc with four boundary points removed, or equivalently, the upper-half-plane with four real points removed. We shall use the latter formulation. The Veneziano amplitude then results from summing over the amplitudes associated with all such punctured half-planes.

Such half-planes can be labelled by four real points x_1, x_2, x_3, x_4. But two such half-planes are conformally equivalent if these parameters have the same cross-ratio. Thus to count each manifold just once, the summation should run only over values of the cross-ratio. This can be achieved by fixing x_1, x_2, x_4 say as $0, 1, \infty$ (a 'gauge') and the summing over x_2. More symmetrically we may

*SERC Advanced Fellow

formally sum over all four parameters and then divide by the infinite volume of the $SL2R$ 'gauge group'. The formalism proposed here has the symmetry of the latter approach, but avoids infinities by replacing the original non-compact integral by the compact integration of a projective form in $(CP^1)^4$. This requires a number of steps: (1) re-interpreting the original real integral as the integration over a real path of a complex form (2) using projective spinors (appropriate in any case because this puts ∞ on an equal footing with other points) (3) using a Pochhammer contour to replace the integration into branch points by compact contour integration round the branch points (4) restoring the symmetry by writing this as an integral in $(CP^1)^4$.

The integral for the amplitude corresponding to the cyclic ordering (1234) (using the 'gauge' choice given above) is

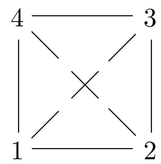
$$\int_0^1 dx_2 \mid x_2 \mid^{k_1.k_2} \mid 1 - x_2 \mid^{k_2.k_3}$$

This becomes (on following these four steps) the spinor integral

$$(2\pi i)^{-2} \oint Dz_1 \wedge Dz_2 \wedge Dz_3 \wedge Dz_4 \ \ [(z_1.z_3)(z_2.z_4)]^{-k_1.k_3}$$

$$\times \ \frac{[(z_1.z_2)(z_3.z_4)]^{-k_1.k_2}}{2i \, \sin \pi(k_1.k_2)} \ \frac{[(z_1.z_4)(z_2.z_3)]^{-k_1.k_4}}{2i \, \sin \pi(k_1.k_4)}$$

which we write diagrammatically as



The 'tachyonic' condition $m^2 = -k_1.k_2 - k_1.k_3 - k_1.k_4 = -2$ is equivalent to this projective integral being well-defined.

This integral may be considered as composed of singular 'propagator' factors of form

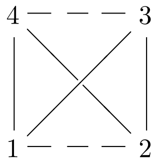
$$\frac{[(z_1.z_2)(z_3.z_4)]^{-k_1.k_2}}{2i \, \sin \pi(k_i.k_2)}$$

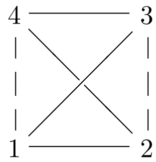
and numerator factors (actually *periods* of these singular factors) of form

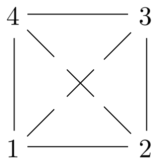
$$[(z_1.z_3)(z_2.z_4)]^{-k_1.k_3}$$

For general values of momenta this distinction is artificial, but when the exponents are integers (the case of interest when studying the twistor analogue) the 'propagators' become logarithmic and the 'numerators' non-singular.

If the external states have $SU(n)$ indices ('quark-antiquark charge' in the original bosonic string theory) then a further coefficient must be specified, namely $\mathrm{tr}(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)$, where Λ_i is the $SU(n)$ matrix on the ith string. The complete amplitude is then given by the sum:

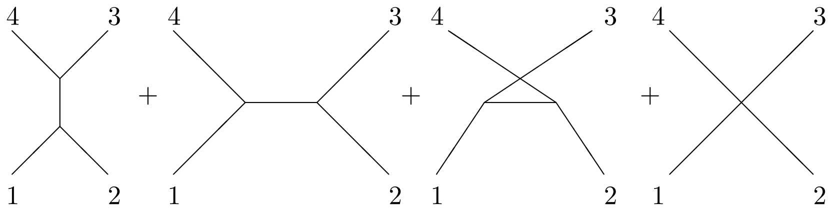
$\mathrm{tr}(\Lambda_1\Lambda_3\Lambda_2\Lambda_4)$ 

$+\ \mathrm{tr}(\Lambda_1\Lambda_2\Lambda_4\Lambda_3)$ 

$+\ \mathrm{tr}(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)$ 

(1)

We now turn attention to field theory in Minkowski space, in fact to pure $SU(2)$ gauge field scattering. The reason for this choice of process is that it turns out to be in a certain sense the simplest to describe as a twistor integral. We shall observe a remarkable parallel to this string-theoretic formula which emerges from this twistorial re-description. To do this we compute this amplitude by standard Feynman rules: this amounts to summing



where the ith state is defined by potential Φ_i^a and $SU(2)$ matrix Λ_i. It turns out that the scattering demonstrates 'helicity conservation', with left and right helicity parts interacting independently. Hence we lose nothing by taking Φ_1^a,Φ_2^a to be self-dual, $i.e.$with 2-spinor representations $\Phi_1^{AA'}$, $\Phi_1^{AA'}$ satisfying

$$\nabla_{BA'}\Phi_1^{AA'} = 0,\ \nabla_{BA'}\Phi_2^{AA'} = 0$$

and the other fields likewise to be anti-self-dual: $i.e.$

$$\nabla_{B'A}\Phi_3^{AA'} = 0,\ \nabla_{B'A}\Phi_4^{AA'} = 0.$$

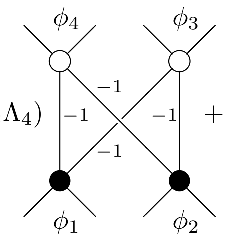
Then the fields are given in 2-spinor form by $\phi_1^{A'B'}(x)$, $\phi_2^{A'B'}(x)$, $\phi_3^{AB}(x)$, $\phi_4^{AB}(x)$, with Fourier transforms $\tilde{\phi}_1^{A'B'}(k_1)$ etc. The general case, in which the interacting fields are not eigenstates of helicity, can be recovered by linearity. The individual Feynman diagrams are gauge-dependent but their sum (by a straightforward but non-trivial computation) may be expressed in the manifestly gauge-invariant form:

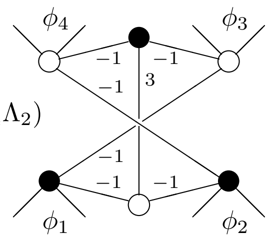
$$\mathrm{tr}(\Lambda_1\Lambda_3\Lambda_2\Lambda_4)\int d^4k_1\ldots d^4k_4\ \frac{\tilde{\phi}_1^{A'B'}(k_1)\tilde{\phi}_3^{AB}(k_3)\tilde{\phi}_{2A'B'}(k_2)\tilde{\phi}_{4AB}(k_4)}{(k_1+k_3)^2\ (k_1+k_4)^2}$$

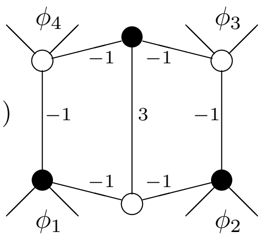
$$+\ \mathrm{tr}(\Lambda_1\Lambda_3\Lambda_4\Lambda_2)\int d^4k_1\ldots d^4k_4\ \frac{\tilde{\phi}_1^{A'B'}(k_1)\tilde{\phi}_2^{AB}(k_2)\tilde{\phi}_{3A'B'}(k_3)\tilde{\phi}_{4AB}(k_4)}{(k_1+k_2)^2\ (k_1+k_3)^2}$$

$$+\ \mathrm{tr}(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)\int d^4k_1\ldots d^4k_4\ \frac{\tilde{\phi}_1^{A'B'}(k_1)\tilde{\phi}_2^{AB}(k_2)\tilde{\phi}_{3A'B'}(k_3)\tilde{\phi}_{4AB}(k_4)}{(k_1+k_2)^2\ (k_1+k_4)^2}$$

Using standard translation techniques (see for instance [8]), each of these terms may be translated into a twistor diagram, yielding the sum:

$\mathrm{tr}(\Lambda_1\Lambda_3\Lambda_2\Lambda_4)$ 

$+\ \mathrm{tr}(\Lambda_1\Lambda_3\Lambda_4\Lambda_2)$ 

$+\ \mathrm{tr}(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)$ 

The definition of these diagrams is specified elsewhere [4, 5, 6] but a brief description can be given. The notation is like Feynman diagram notation, in that all the vertices represent variables to be integrated out. But these variables are twistors or dual twistors (corresponding to black or white vertices respectively) and the integration is compact contour integration. The external fields appear in the standard twistor representation, *i.e.* as first cohomology group elements. The 'propagator' lines connecting the vertices are simple singular factors. The whole structure is manifestly finite and is also manifestly conformally invariant.

However, the aspect of the integration that concerns us here is the analogy with the sum of spinor integrals (1).

The essential point is that in each diagram the form to be integrated is simply the product of the external fields and a natural volume form on the total space. The contour over which it is to be integrated is dictated by the presence of lines labelled (-1). These lines define logarithmic factors in the integral, and the contour can be regarded as a higher-dimensional Pochhammer contour which winds round the branch curves they define. Note that the corresponding 'numerators' are just unity in this case, and also that these logarithmic factors connect the external states in the same cyclic order as in the trace yielding the $SU(2)$ coefficient. Thus the sum of the twistor diagrams is of just the same form as the sum (1) of spinor integrals. We know that these spinor integrals can be derived as integrals over a parameter space of amplitudes arising from a conformal field theory. This suggests that it may be possible to derive the analogous twistor integral from a CFT4 principle, instead of producing it by translation from field theory.

The pure $SU(2)$ gauge field scattering integral is special because only for it does the twistor diagram representation reduce to the integration of a volume form, making the analogy with string theory particularly close. To describe the interaction of fields with spin other than 1, one must integrate certain very simple rational functions rather than pure volume. However, there seems no reason why a generalisation encompassing this feature could not follow from a CFT4 principle.

References

[1] A. P. Hodges, R. Penrose, M. A. Singer *Phys. Letters* B **216** 48–52 (1989)

[2] R. Penrose *this volume*

[3] M. A. Singer *this volume*

[4] A. P. Hodges, *Physica* **114A** 157–175 (1982)

[5] A. P. Hodges, *Proc. R. Soc. Lond.* **A 397** 341–374 (1985)

[6] A. P. Hodges, *Proc. R. Soc. Lond.* **A 397** 375–396 (1985)

[7] M. B. Green, J. H. Schwarz, E. Witten *Superstring theory* (Cambridge University Press, 1987)

[8] A. P. Hodges, *Proc. R. Soc. Lond.* **A 386** 185-210 (1983)

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